Research and analysis of the various solutions provided for the Brachistochrone problem

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Abstract

The Brachistochrone problem, meaning in greek "shortest time", is the question regarding what is the shape of the path to slide a point mass between two arbitrary points with a height difference in the shortest time possible, while considering only the action of a constant gravitational force applied on it. This project will present the problem, and the solutions provided for the problem by Johann Brenoulli, and through calculus of variations.

Introduction

The problem was first suggested by Johann Bernoulli in 1696, stated as:

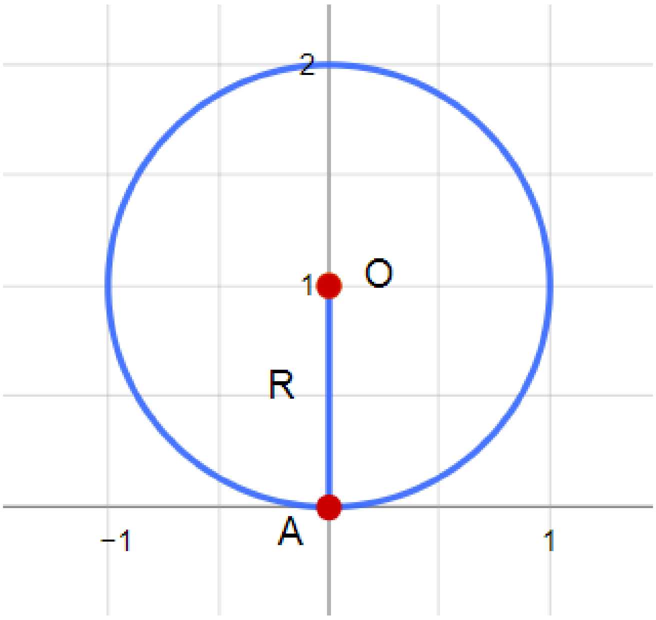
'Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time'

When a person is first addressed with this problem, the intuitive answer is that the path is a straight line, assuming the shortest path naturally provides the shortest time. This answer is of course incorrect because while the mass has a short distance to cover during its decline the acceleration is very low because of the constant path incline and so the mass doesn't gain velocity fast enough. Galileo, prior to Bernoulli, while dealing with a similar problem proposed a solution of the path of a circle, which is a descent solution, but it's not the optimal solution. The optimal solution is known to be the shape of a cycloid, the path of a point of a rolling circle on a straight surface. This solution may be computed with several methods, including Johan Bernoulli's method using analogy to the motion of light with Snell's law, and by calculus of variations while minimizing the cost function of the time of descent using the Euler-Lagrange equations. The problem was also solved by Jakob Bernoulli, Johan's brother, and by Isaac Newton, who formed his solution anonymously.

The path of a Cycloid

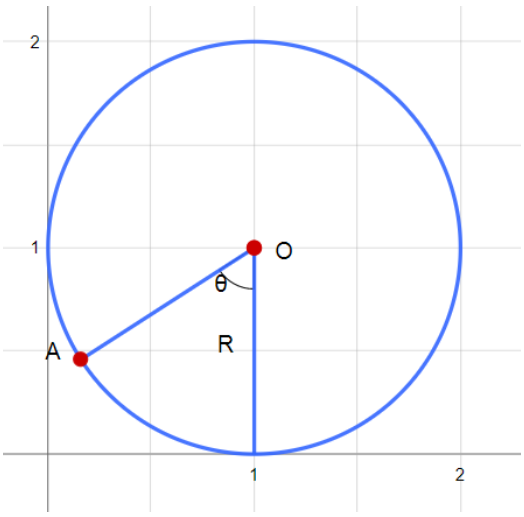
In order to present the derivations of the Brachistochrone solutions it is first required to define the shape of the solution path, the Cycloid.

Given a circle with radius R, rolling on a straight line on the x axis. It is desired to form the equations of the path of a given point on the circle, initially located at $A(0,0)$.



During the rolling of the circle, point A moves around the center O. Define θ as the angle between the segment OA, and the initial segment when $A(0,0)$. The length of OA is R as it is the radius of the circle. The center O position changes with respect to θ as the circle performs a pure roll by the following equations:

$$x_O = \theta R, \qquad y_O = R = const$$

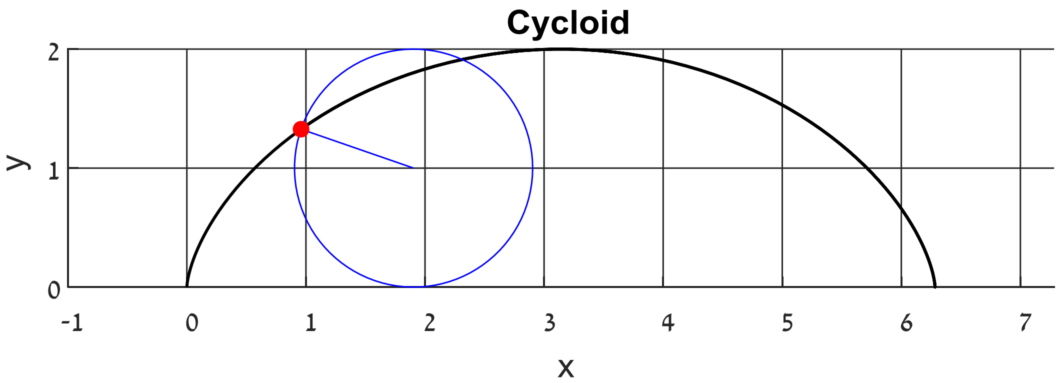


Using trigonometric relations it is derived that:

$$x_A = x_O - Rsin(\theta) = \theta R - Rsin(\theta), \qquad y_A = R - Rcos(\theta)$$

Therefore, the equations of a cycloid are:

$$\begin{cases} x = R(\theta - \sin\theta) \\ y = R(1 - \cos\theta) \end{cases}$$



Bernoulli's Solution

Johan Bernoulli solved the Brachistochrone problem using an analogy to the movement of a beam of light traveling through a varying medium. The proof assumes Snell's law, so first it is required to derive it:

Snell's law states that while a beam of light travels between one medium to another it will deflect according to the following relation:

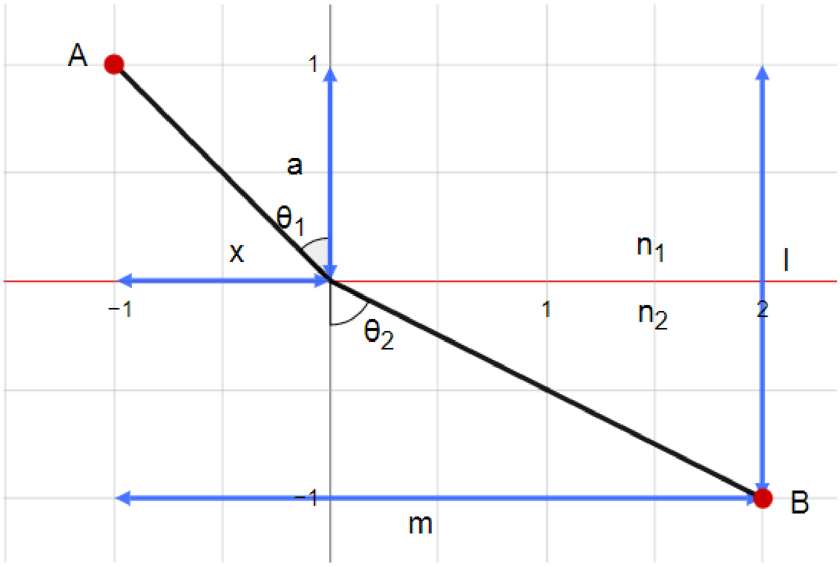
$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

While θ is the angle between the beam of light and the perpendicular to the medium transition line, and n is the refractive index - the ratio between the speed of light in vacuum and the speed of light in the given medium.

$$n = \frac{c}{v}$$

Snell's law is the implementation of Fermat's principle, which at the time was an ampirical law that stated light would find the path to travel between two given points at the minimal time. At later dates, with a better understanding of the nature of light Fermat's principle was proven using Maxwell's equations of electromagnetism, and by the wave-particle duality using quantum mechanics.

Given points A, and B which lie in different mediums n_1, and n_2 accordingly. Define the horizontal length m between A, and B, and the vertical length l. The length between A and the medium transition line is a. Mark x as the horizontal length between A and the point of transition between the mediums, which is unknown.



The velocity in the mediums n_1, and n_2, are v_1, and v_2 accordingly.

The distance between A, and the point of transition:

$$d_1 = \sqrt{x^2 + a^2}$$

The distance between B, and the point of transition:

$$d_2 = \sqrt{(l-a)^2 + (m-x)^2}$$

Since the light velocity is constant in each medium, the time of travel is:

$$t_1 = \frac{d_1}{v_1},\ t_2 = \frac{d_2}{v_2}$$

Thus, the total time of travel between A, and B is:

$$t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2} = \frac{\sqrt{x^2 + a^2}}{v_2} + \frac{\sqrt{(l-a)^2 + (m-x)^2}}{v_2}$$

Using Fermat's principle, we wish to minimize the time as function of x:

$$\frac{dt}{dx} = 0$$

$$\frac{dt}{dx} = \frac{2x}{2v_1\sqrt{x^2 + a^2}} - \frac{2(m-x)}{2v_2\sqrt{(l-a)^2 + (m-x)^2}} = 0$$

$$\frac{x}{v_1 d_1} - \frac{(m-x)}{v_2 d_2} = 0$$

$$\frac{x}{d_1} = \sin\theta_1\,, \qquad \frac{m-x}{d_2} = \sin\theta_2$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$$

We've received Snell's law:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

By applying the relation $n = \frac{c}{v}$ the better known equation is obtained:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

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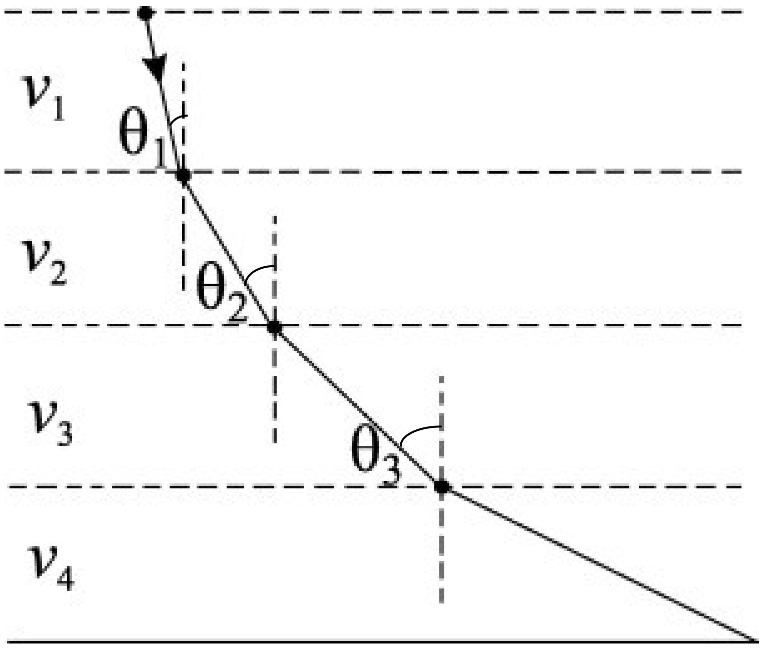
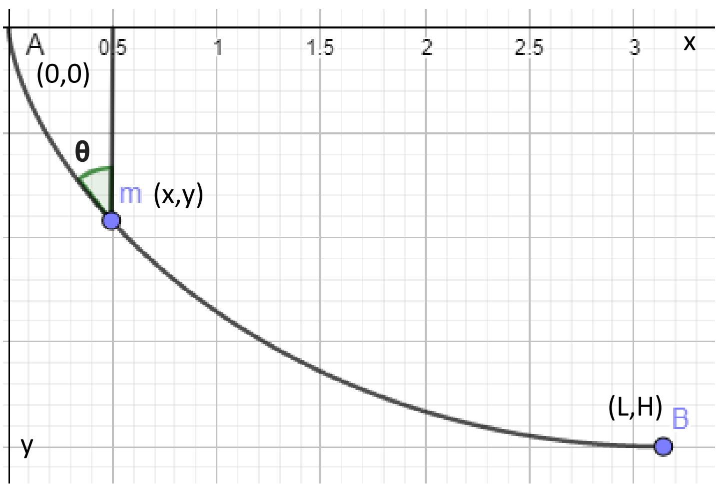
Bernoulli used an analogy between the motion of the point mass on the surface, and a motion of a light beam between infinitely many varying mediums.



Assume a point mass travels from point A, to B using only the gravitational force. Set a Cartesian coordinate system such that A is located at $(0,0)$, and B at known (L, H) beneath point A. θ is the angle between the tangent to the surface and y axis.



Since the only force applied on the mass is gravity, the total energy is conserved.

$$E = V + T = -mgy + \frac{mv^2}{2} = const$$

$$E_A = 0$$

$$-mgy + \frac{mv^2}{2} = 0$$

$$\Rightarrow v = \sqrt{2gy}$$

From Snell's law:

$$\frac{\sin\theta}{v} = const$$

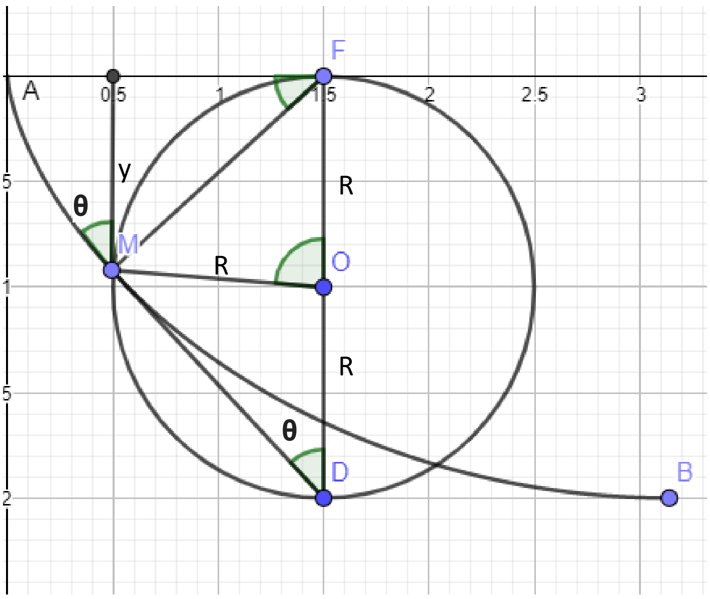
$$\frac{\sin\theta}{\sqrt{2gy}} = const$$

Squaring both sides and adding g to the constant:

$$\frac{\sin^2\theta}{y} = const$$

This relation represents the differential equation of a cycloid. To show how, a geometric proof by mathematician Mark Levi is provided.

Consider the following sketch:



Since the cycloid is created from a rolling circle with a radius R, at any given time F is the instant center of rotation of the circle, so every point on the circle rotates around F at that moment and performs a circular motion around that point. M moves in a circular motion with respect to F at any given time, so it's velocity is perpendicular to the line MF. The velocity vector is in the same plane as the surface the mass slides on, so the tangent line of the surface at any given moment is perpendicular to MF. Continue the tangent on a straight line until it reaches the circle on point D, such that $\sphericalangle FMD = 90°$. A circumferential angle that equals $90°$ lies on the diameter, so FD is the diameter of the circle.

Define the angle $\sphericalangle \mathrm{FDM} = \theta$. θ is a circumferential angle, so the central angle that lies on the same arc, $\sphericalangle \mathrm{FOM} = 2\theta$. θ is also the angle of the mass because they're parallel angles.

The angle between a chord in the circle to the tangent of the circle is the same as the circumferential angle that lies on this chord from the other side, so:

$$\sphericalangle \mathrm{FDM} = \sphericalangle \mathrm{MFA} = \theta$$

Using the law of sines:

$$\frac{MF}{\sin(\sphericalangle \mathrm{FDM})} = \frac{DF}{\sin(\sphericalangle \mathrm{FMD})}$$

$$\frac{MF}{\sin\theta} = \frac{2R}{\sin 90°}$$

$$\Rightarrow MF = 2R\sin\theta$$

$$\frac{y}{MF} = \sin(\sphericalangle \mathrm{MFA})$$

$$\frac{y}{2R\sin\theta} = \sin\theta$$

$$\frac{\sin^2\theta}{y} = \frac{1}{2R} = const$$

We've received the same equation that was derived by Bernoulli. This equation represents the cycloid equation:

$$\begin{cases} x = R(2\theta - \sin 2\theta) \\ y = R(1 - \cos 2\theta) \end{cases}$$

While θ is the angle between the tangent to the surface and y axis, it is half the angle of the circle's rotation. R is the radius of the circle:

$$R = \frac{H}{2}$$

Calculus of variations

Calculus of variations is a branch in calculus that deals with finding the minima of functionals. A functional is a function which its variables are functions as well. An example for a functional is an integral:

$$F\big(y(x)\big) = \int_{x_1}^{x_2} y(\xi)\, d\xi$$

The functional used in variational calculus is called a cost function:

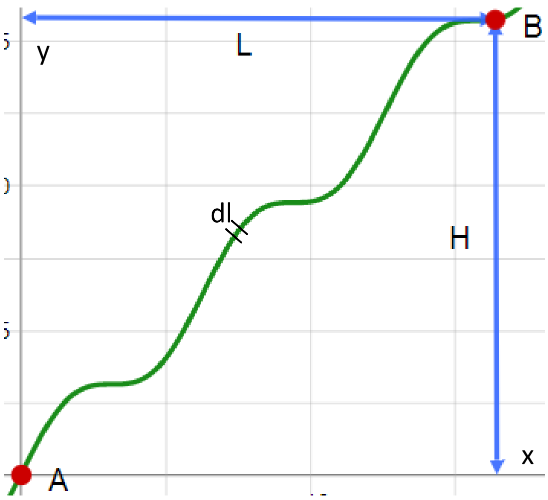
$$J(y) = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big) dx$$

It is required to find $y(x)$ such that the solution to $J(y)$ is minimal. In order to find the minimal solution, F must satisfy the Euler-Lagrange equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

Example - Shortest path problem:

It is required to find the path with the shortest possible length between two points A, and B. Define the coordinates system so that $A = (0,0),\ B = (L, H)$.



The differential length of the path:

$$dl = \sqrt{dx^2 + dy^2}$$

Deriving the relations between dx, and dy:

$$dx = dx \cdot \frac{dy}{dy} = \frac{dx}{dy} \cdot dy = x'(y) \cdot dy$$

It is desired to find a path with minimal length, so it is the cost function:

$$J = l(x) = \int_{y_1}^{y_2} F\big(y, x(y), x^{'}(y)\big)dy$$

$$\min_{y} l = \int_{\mathrm{y}} \mathrm{dl} = \int_{\mathrm{y=0}}^{\mathrm{y=H}} \sqrt{dx^2 + dy^2} = \int_{\mathrm{y=0}}^{\mathrm{y=H}} \sqrt{\left(x^{'}(y)\right)^2 + 1}\,\mathrm{dy}$$

Therefore:

$$F\big(y, x(y), x^{'}(y)\big) = \sqrt{\left(x^{'}(y)\right)^2 + 1}$$

Applying Euler-Lagrange equation:

$$\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial x^{'}} = 0$$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\sqrt{\left(x^{'}(y)\right)^2 + 1} = 0$$

$$\frac{\partial F}{\partial x^{'}} = \frac{\partial}{\partial x^{'}}\sqrt{\left(x^{'}(y)\right)^2 + 1} = \frac{x^{'}(y)}{\sqrt{\left(x^{'}(y)\right)^2 + 1}}$$

$$\frac{\partial F}{\partial x} - \frac{d}{dy}\frac{\partial F}{\partial x^{'}} = \frac{d}{dy}\frac{x^{'}(y)}{\sqrt{\left(x^{'}(y)\right)^2 + 1}} = 0$$

$$\Rightarrow \frac{x^{'}(y)}{\sqrt{\left(x^{'}(y)\right)^2 + 1}} = c = const$$

Squaring both sides of the equation:

$$\frac{\left(x^{'}(y)\right)^2}{\left(x^{'}(y)\right)^2 + 1} = c^2$$

$$\left(x^{'}(y)\right)^2 = \mathrm{c}^2\left(x^{'}(y)\right)^2 + \mathrm{c}^2$$

$$\left(x^{'}(y)\right)^2 = \frac{c^2}{1 - c^2}$$

$$x^{'}(y) = \sqrt{\frac{c^2}{1 - c^2}} = a = const$$

$$\Longrightarrow x(y) = a \cdot y + b$$

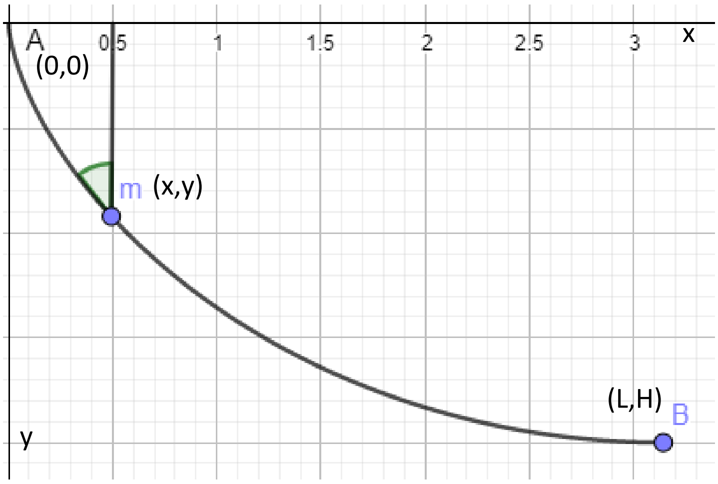
We've obtained a linear function, so the path that provides the shortest possible length is in fact a straight line. Of course this is trivial because it is obvious that the path with the shortest length between two points on a cartesian plane is a straight line.

Applying the initial, and the terminal conditions:

$$x(y=0)=0, \qquad x(y=H)=L$$

$$x(y)=\frac{L}{H}y$$

Brachistochrone solution using variational calculus



The differential length of the path:

$$dl=\sqrt{dx^2+dy^2}$$

Deriving the relations between dx, and dy:

$$dx=dx\cdot\frac{dy}{dy}=\frac{dx}{dy}\cdot dy=x'(y)\cdot dy$$

From conservation of energy:

$$v=\sqrt{2gy}$$

It is required to find the path of minimal time, so the time of descent is our cost function:

$$\min_y t=\int_l \frac{dl}{v}=\int_{y=0}^{y=H}\frac{\sqrt{\left(x'(y)\right)^2+1}}{\sqrt{2gy}}dy=\frac{1}{\sqrt{2g}}\int_{y=0}^{y=H}\sqrt{\frac{\left(x'(y)\right)^2+1}{y}}dy$$

$$F\big(y,x(y),x'(y)\big)=\sqrt{\frac{\left(x'(y)\right)^2+1}{y}}$$

Euler-Lagrange equations:

$$\frac{\partial F}{\partial x}-\frac{d}{dy}\frac{\partial F}{\partial x'}=0$$

$$\frac{\partial F}{\partial x}=0$$

$$\frac{\partial F}{\partial x'}=\frac{\partial}{\partial x'}\sqrt{\frac{\left(x'(y)\right)^2+1}{y}}=\frac{1}{\sqrt{y}}\frac{x'(y)}{\sqrt{\left(x'(y)\right)^2+1}}$$

$$\frac{\partial F}{\partial x}-\frac{d}{dy}\frac{\partial F}{\partial x'}=\frac{d}{dy}\frac{x'(y)}{\sqrt{y\left(\left(x'(y)\right)^2+1\right)}}=0$$

$$\frac{x'(y)}{\sqrt{y\left(\left(x'(y)\right)^2+1\right)}}=const$$

Square both sides:

$$\frac{\left(x'(y)\right)^2}{y\left(\left(x'(y)\right)^2+1\right)}=\frac{1}{2a}=const$$

$$2a\left(x'(y)\right)^2=y\left(x'(y)\right)^2+y$$

$$\left(x'(y)\right)^2(2a-y)=y$$

$$x'(y)=\sqrt{\frac{y}{2a-y}}$$

$$x(y)=\int_{y=0}^{y}\sqrt{\frac{y}{2a-y}}\,dy$$

Use parameter substitution:

$$y=a-a\cos\theta$$

$$dy=a\sin\theta\,d\theta$$

$$x(\theta)=\int_{\theta=0}^{\theta}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}\,a\sin\theta\,d\theta=a\int_{\theta=0}^{\theta}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta\,d\theta$$

$$=a\int_{\theta=0}^{\theta}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sqrt{1-\cos^2\theta}\,d\theta=a\int_{\theta=0}^{\theta}\sqrt{\frac{(1-\cos\theta)(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}}\,d\theta$$

$$= a \int_{\theta=0}^{\theta} \sqrt{(1-\cos\theta)^2}\, d\theta = a \int_{\theta=0}^{\theta} (1-\cos\theta)\, d\theta = a(\theta - \sin\theta)$$

$$\begin{cases} x(\theta) = a(\theta - \sin\theta) \\ y(\theta) = a(1-\cos\theta) \end{cases}$$

We've obtained the equations of a cycloid.

For $\theta = 0$: $x(0) = 0,\ y(0) = 0$

For $\theta = \theta_f$: $x(\theta_f) = L,\ y(\theta_f) = H$

Time of movement on a cycloid

We would like to compute the time it takes for the point mass to move from the top of the cycloid, at $y = 0$, to the bottom at $y = H$.

The time differential:

$$dt = \frac{dl}{v}$$

The cycloid equations:

$$\begin{cases} x(\theta) = a(\theta - \sin\theta) \\ y(\theta) = a(1-\cos\theta) \end{cases}$$

$$dx = (a - a\cos\theta)d\theta, \qquad dy = a\sin\theta\, d\theta$$

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}\, d\theta = a\sqrt{2 - 2\cos\theta}\, d\theta$$

$$v = \sqrt{2gy} = \sqrt{2ga(1-\cos\theta)} = \sqrt{ga(2-2\cos\theta)}$$

$$dt = \frac{dl}{v} = \frac{a\sqrt{2-2\cos\theta}}{\sqrt{ga(2-2\cos\theta)}} d\theta = \sqrt{\frac{a}{g}}\, d\theta$$

$$t = \int_{\theta=0}^{\theta} \sqrt{\frac{a}{g}}\, d\theta = \sqrt{\frac{a}{g}}\,\theta$$

The mass reaches the bottom of the cycloid at $\theta = \pi$

$$t = \pi\sqrt{\frac{a}{g}}$$

a, is the radius of the rolling circle, so:

$$a = \frac{H}{2}$$

If the mass starts to slide from an intermediate point θ_0:

$$y_0 = a(1 - \cos\theta_0)$$

$$x_0 = a(\theta_0 - \sin\theta_0)$$

$$E = E_A = const$$

$$\frac{mv^2}{2} - mgy = -mgy_0$$

$$v = \sqrt{2g(y - y_0)} = \sqrt{2ga((1 - \cos\theta) - (1 - \cos\theta_0))}$$

$$= \sqrt{2ga(\cos\theta_0 - \cos\theta)}$$

The time differential:

$$dt = \frac{dl}{v} = \frac{a\sqrt{2 - 2\cos\theta}}{\sqrt{2ga(\cos\theta_0 - \cos\theta)}} d\theta = \sqrt{\frac{a}{g}} \sqrt{\frac{1 - \cos\theta}{\cos\theta_0 - \cos\theta}} d\theta$$

$$t = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\theta} \sqrt{\frac{1 - \cos\theta}{\cos\theta_0 - \cos\theta}} d\theta$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$$

$$t = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\theta} \sqrt{\frac{1 - 2\cos^2\frac{\theta}{2} + 1}{2\cos^2\frac{\theta_0}{2} - 1 - 2\cos^2\frac{\theta}{2} + 1}} d\theta = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\theta} \sqrt{\frac{1 - \cos^2\frac{\theta}{2}}{\cos^2\frac{\theta_0}{2} - \cos^2\frac{\theta}{2}}} d\theta$$

$$= \sqrt{\frac{a}{g}} \int_{\theta_0}^{\theta} \frac{\sin\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta_0}{2} - \cos^2\frac{\theta}{2}}} d\theta$$

Parameter substitution:

$$u = \frac{\cos\frac{\theta}{2}}{\cos\frac{\theta_0}{2}}$$

$$du = -\frac{\frac{1}{2}\sin\frac{\theta}{2}}{\cos\frac{\theta_0}{2}} d\theta$$

$$t = -\sqrt{\frac{a}{g}} \int_{u_0}^{u} \frac{\sin\frac{\theta}{2}}{\sqrt{\cos^2\frac{\theta_0}{2} - u^2\cos^2\frac{\theta_0}{2}}} \frac{\cos\frac{\theta_0}{2}}{\frac{1}{2}\sin\frac{\theta}{2}} du = -\sqrt{\frac{a}{g}} \int_{u_0}^{u} \frac{2\cos\frac{\theta_0}{2}}{\cos\frac{\theta_0}{2}\sqrt{1 - u^2}} du$$

$$= -2\sqrt{\frac{a}{g}}\int_{\theta_0}^{\theta}\frac{1}{\sqrt{1-u^2}}du \ = -2\sqrt{\frac{a}{g}}\cdot \arcsin u$$

$$= -2\sqrt{\frac{a}{g}}\cdot \arcsin\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta_0}{2}}\Bigg|_{\theta_0}^{\theta} = -2\sqrt{\frac{a}{g}}\cdot \arcsin\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta_0}{2}} + 2\sqrt{\frac{a}{g}}\cdot\frac{\pi}{2}$$

$$t = -2\sqrt{\frac{a}{g}}\cdot \arcsin\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta_0}{2}} + \pi\sqrt{\frac{a}{g}}$$

At the bottom of the cycloid: $\theta = \pi$

$$t = -2\sqrt{\frac{a}{g}}\cdot \arcsin\frac{\cos\frac{\pi}{2}}{\cos\frac{\theta_0}{2}} + \pi\sqrt{\frac{a}{g}} = -2\sqrt{\frac{a}{g}}\cdot \arcsin\frac{0}{\cos\frac{\theta_0}{2}} + \pi\sqrt{\frac{a}{g}}$$

$$t = \pi\sqrt{\frac{a}{g}}$$

From any point the mass starts sliding on the cycloid it will reach the bottom of the cycloid at the same time. A mass placed at the top of the cycloid, and a mass placed half way through will reach the bottom together.

Comparison to a straight line

In this paper it was proven mathematically that the time of descent of a cycloid is the minimal possible of all path shapes, we'd like to justify this proof by comparing it to other paths.

The equation of a straight line is linear:

$$y(x) = ax + b$$

providing the initial, and terminal conditions:

$$y(x) = \frac{H}{L}x$$

$$dy = \frac{H}{L}dx$$

The time differential:

$$dt = \frac{dl}{v}$$

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{1 + \frac{H^2}{L^2}}\, dx$$

$$v = \sqrt{2gy} = \sqrt{2g\frac{H}{L}x}$$

$$dt = \frac{dl}{v} = \frac{\sqrt{1 + \frac{H^2}{L^2}}}{\sqrt{2g\frac{H}{L}x}} dx = \sqrt{\frac{1 + \frac{H^2}{L^2}}{2g\frac{H}{L}}} \frac{1}{\sqrt{\mathrm{x}}} \mathrm{dx}$$

$$t = \sqrt{\frac{1 + \frac{H^2}{L^2}}{2g\frac{H}{L}}} \int_{x=0}^{\mathrm{x}} \frac{1}{\sqrt{\mathrm{x}}} \mathrm{dx} = 2\sqrt{\frac{1 + \frac{H^2}{L^2}}{2g\frac{H}{L}}} \sqrt{x}$$

At the end of the path, $x = L$:

$$t_{line} = 2\sqrt{\frac{1 + \frac{H^2}{L^2}}{2g\frac{H}{L}}} \sqrt{L} = 2\sqrt{\frac{L^2 + H^2}{2gH}}$$

The descent time for a cycloid:

$$t_{cycloid} = \pi\sqrt{\frac{a}{g}} = \pi\sqrt{\frac{H}{2g}}$$

For a full cycloid where the mass is released from the top of the cycloid:

$$L = \pi a = \frac{\pi H}{2}$$

$$t_{line} = 2\sqrt{\frac{L^2 + H^2}{2gH}} = 2\sqrt{\frac{\left(\frac{\pi H}{2}\right)^2 + H^2}{2gH}} = \sqrt{\frac{\pi^2 + 4}{2}}\sqrt{\frac{H}{g}}$$

$$\frac{t_{line}}{t_{cycloid}} = \frac{\sqrt{\frac{\pi^2 + 4}{2}}}{\frac{\pi}{\sqrt{2}}} \cong 1.185$$

The descent time of a straight line is indeed slower than the cycloid path.

Gravitational potential inside, and outside the earth

It is required to obtain an expression for the gravitational potential at a point inside the earth. The derivation is done using Gauss's law for gravitational fields:

$$\oint_S \underline{g} \cdot d\underline{S} = -4\pi G M_{in}$$

While $\underline{S}$ is the area vector of a closed surface, M_{in} is the total mass enclosed within the surface, and $\underline{g}$ is the gravitational field.

First, deriving the gravitational field at a point with radius r outside the earth. The earth is assumed to be a perfect solid sphere with radius R, and density ρ. The mass of the earth:

$$M = \int_V \rho \, dV = \frac{4}{3}\pi R^3 \rho$$

The closed surface chosen is a sphere with radius r, the surface area:

$$\underline{S} = 4\pi r^2 \underline{\hat{r}}$$

Since there is radial symmetry, the integral is simplified as:

$$\oint_S \underline{g} \cdot d\underline{S} = \underline{g} \cdot \underline{S} = \left(g\underline{\hat{r}}\right) \cdot \left(4\pi r^2 \underline{\hat{r}}\right) = 4\pi g r^2$$

Applying Gauss's law:

$$4\pi g r^2 = -4\pi G M_{in}$$

$r > R$, so the mass enclosed within $\underline{S}$ is the entire mass of the earth:

$$g = -\frac{GM}{r^2}$$

From the radial symmetry:

$$\underline{g} = -\frac{GM}{r^2}\underline{\hat{r}}$$

The gravitational potential at a point $r > R$:

$$V(r) = -\int \underline{g} \cdot d\underline{r} = \int \frac{GM}{r^2} dr = -\frac{GM}{r} + c$$

It is customary to define the potential such that:

$$\lim_{r \to \infty} V(r) = 0$$

So $c = 0$

Overall, the gravitational potential outside a solid sphere:

$$V(r) = -\frac{GM}{r}$$

The gravitational field inside the earth is computed using Gauss's law as well. For $r < R$ the surface S is chosen a sphere with radius r:

$$\underline{S} = 4\pi r^2 \underline{\hat{r}}$$

Since the surface is inside the earth, the mass inside is proportional to the volume enclosed within the surface:

$$M_{in} = \frac{4}{3}\pi r^3 \rho$$

The ratio between the total mass and the enclosed mass:

$$\frac{M_{in}}{M} = \frac{\frac{4}{3}\pi r^3 \rho}{\frac{4}{3}\pi R^3 \rho} = \frac{r^3}{R^3}$$

$$\Rightarrow M_{in} = \frac{r^3}{R^3} M$$

Gauss's law:

$$\oint_S \underline{g} \cdot d\underline{S} = -4\pi G M_{in}$$

$$4\pi g r^2 = -4\pi G \frac{r^3}{R^3} M$$

$$g = -G \frac{r}{R^3} M$$

From the radial symmetry:

$$\underline{g} = -\frac{GM}{R^3} r \underline{\hat{r}}$$

It is obtained that the gravitational field inside a solid sphere is linearly proportional to the radius, similar to harmonic oscillators.

The potential inside the earth at a point $r < R$:

$$V(r) = -\int \underline{g} \cdot d\underline{r} = \int \frac{GM}{R^3} r \, dr = \frac{GM}{2R^3} r^2 + c$$

Since the gravitational potential is required to be continuous, it is equal to the potential outside the sphere on the border at $r = R$:

$$V(R) = -\frac{GM}{R}$$

$$\Rightarrow c = -\frac{3}{2}\frac{GM}{R}$$

So the gravitational potential inside the sphere:

$$V(r) = \frac{GM}{2R}\left(\frac{r^2}{R^2} - 3\right)$$

Overall, the potential of the earth:

$$V(r) = \begin{cases} \frac{GM}{2R}\left(\frac{r^2}{R^2} - 3\right), & r < R \\ -\frac{GM}{r}, & r \geq R \end{cases}$$

Snell's Law derivation for polar coordinates

Using Fermat's principle of minimum time, it is desired to compute Snell's law while the refractive index changes radially on an axis-symmetric sphere, so in polar coordinates:

$$n = f(r)$$

Given two mediums, one outside a sphere with radius $r = a$, and the second inside the sphere. The refractive index is therefore:

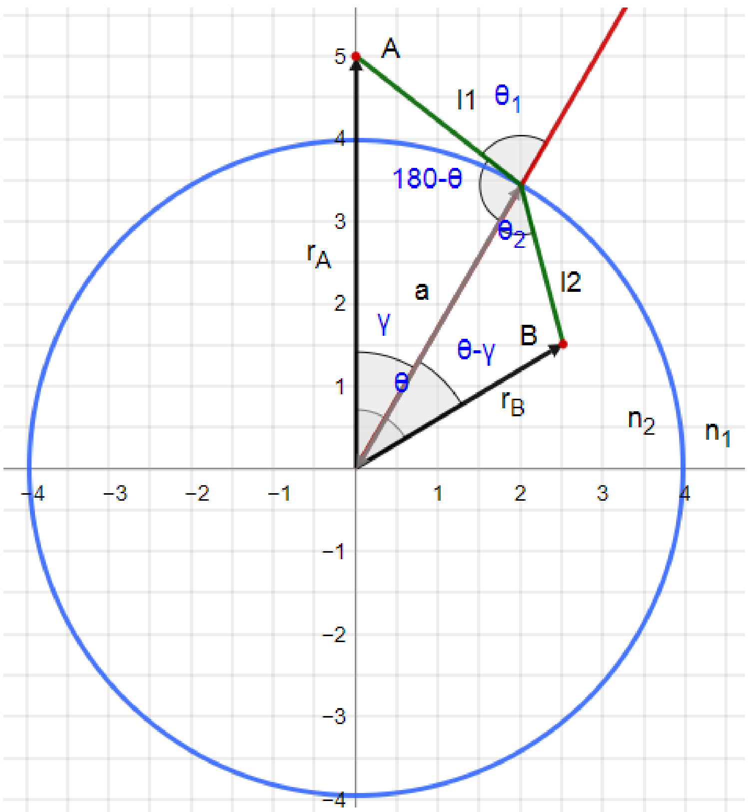
$$n(r) = \begin{cases} n_1, & r > a \\ n_2, & r < a \end{cases}$$

Choosing two arbitrary points: A is outside the sphere, and B is inside the sphere.

$$r(\theta_A) = r_A, \qquad r(\theta_B) = r_B$$

$$\theta = \theta_B - \theta_A$$

It is required to find the point at which the light would choose to pass from n_1 to n_2 in order to travel from point A to point B at the minimum possible time.



The light would travel from point A to the sphere intersection point an angular distance of γ, the linear distance traveled over this angle is obtained via the cosine theorem:

$$l_1{}^2 = a^2 + r_A{}^2 - 2ar_A \cos\gamma$$

Similarly, for the distance from the intersection to B:

$$l_2{}^2 = a^2 + r_B{}^2 - 2ar_B\cos(\theta - \gamma)$$

The light travels in a given medium at a velocity of:

$$v = \frac{c}{n}$$

While c is the speed of light in vacuum.

The time that takes the light to cover the distances:

$$t_1 = \frac{l_1}{v_1} = \frac{\sqrt{a^2 + r_A{}^2 - 2ar_A\cos\gamma}}{v_1}, \qquad t_2 = \frac{l_2}{v_2} = \frac{\sqrt{a^2 + r_B{}^2 - 2ar_B\cos(\theta - \gamma)}}{v_2}$$

The total time of travel:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + r_A{}^2 - 2ar_A\cos\gamma}}{v_1} + \frac{\sqrt{a^2 + r_B{}^2 - 2ar_B\cos(\theta - \gamma)}}{v_2}$$

Applying Fermat's principle of minimum time:

$$\frac{dt}{d\gamma} = 0$$

$$\frac{dt}{d\gamma} = \frac{ar_A\sin\gamma}{v_1\sqrt{a^2 + r_A{}^2 - 2ar_A\cos\gamma}} - \frac{ar_B\sin(\theta - \gamma)}{v_2\sqrt{a^2 + r_B{}^2 - 2ar_B\cos(\theta - \gamma)}} = 0$$

$$\frac{r_A\sin\gamma}{v_1 l_1} - \frac{r_B\sin(\theta - \gamma)}{v_2 l_2} = 0$$

Using the sine theorem:

$$\frac{l_1}{\sin\gamma} = \frac{r_A}{\sin(\pi - \theta_1)} = \frac{r_A}{\sin\theta_1}, \qquad \frac{l_2}{\sin(\theta - \gamma)} = \frac{r_B}{\sin\theta_2}$$

$$\frac{r_A\sin\theta_1}{v_1 r_A} - \frac{r_B\sin\theta_2}{v_2 r_B} = 0$$

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

$$\Longrightarrow n_1\sin\theta_1 = n_2\sin\theta_2$$

This result is the same as the law derived in linear coordinates.

Bouguer's Law derivation:

$$n(r)\cdot r\sin\theta = const$$

$$\frac{r\sin\theta}{v(r)} = const$$

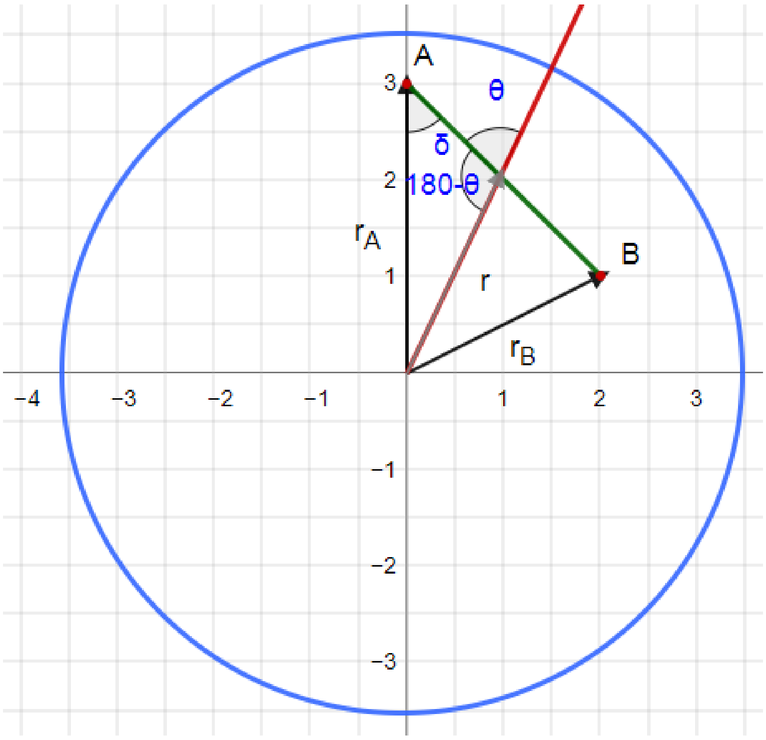
Assume homogenous spherical medium, the velocity inside the sphere is constant:

$$v(r) = const$$

It is only required to prove that:

$$r \sin\theta = const$$

Since there is no refraction while $n = const$ the fastest route would be a straight line, and that is the path in which the light travels.



The angle between the trajectory of the light and the initial radius vector r_A is constant.

$$\delta = const$$

Using the sine theorem for every $r, \theta(r)$ throughout the course between points A, and B:

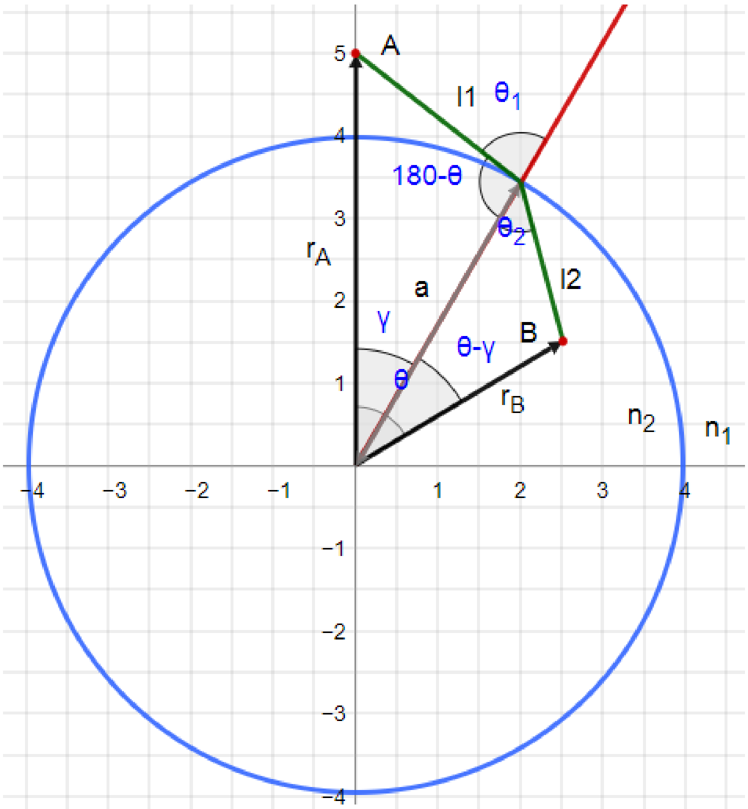
$$\frac{\sin\delta}{r} = \frac{\sin(180 - \theta)}{r_A}$$

$$r \sin\theta = r_A \sin\delta = const$$

Since $n(r)$ is constant as long as the movement is a straight line:

$$n(r) \cdot r \sin\theta = const$$

Assuming there is a medium change between points A, and B:



During the movement of the light through l_1, and through l_2 there is no medium change so it has been showed that the equation holds for these parts of the course.

At the point of the refraction it was proven that Snell's law applies:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Since the radius is the same on that point it can be multiplied on both sides of the equation:

$$n_1 a \cdot \sin \theta_1 = n_2 a \cdot \sin \theta_2$$

Since the equation is true on the refraction points and also between refractions it applies throughout all of the movement between points A, and B.

So, overall:

$$n(r) \cdot r \sin \theta = const$$

Solving the Brachistochrone problem for a spherical earth

Assume a spherical earth with a gravitational field:

$$\underline{g} = -\frac{M_{\oplus}G}{r^2}\underline{\hat{r}}$$

The center of the earth in ECI coordinates is at $\underline{r_O} = [0\ 0\ 0]^T$.

It is required to find the course from point $A(x_A, y_A, z_A)$ to point $B(x_B, y_B, z_B)$ which a point mass would travel at the shortest time while applied only a gravitational force directed to $\underline{r_O}$.

$$r_A \geq r_B$$

Since the earth is assumed to be a perfect sphere, and the gravity is assumed to be only dependent on $\underline{r}$, there exists a coordinate system where A, and B both lie on the same plane. So using polar coordinates:

$$A(r_A, \theta_A), \qquad B(r_B, \theta_B)$$

The transformation from cartesian to polar coordinates:

$$x = r\cos\theta\,, y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta\,, \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta\,, \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$dl_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial x}{\partial r}\right)^2}\, dr = \sqrt{\cos^2\theta + \sin^2\theta}\, dr = dr$$

$$dl_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial x}{\partial \theta}\right)^2}\, d\theta = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\, d\theta = rd\theta$$

$$dl = \sqrt{{dl_r}^2 + {dl_\theta}^2} = \sqrt{dr^2 + r^2 d\theta^2} = \sqrt{dr^2 + r^2\left(\frac{d\theta}{dr}\right)^2 dr^2} = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

The kinetic energy of the mass during its course:

$$T = \frac{1}{2}mv^2$$

The gravitational potential energy of the mass during its course:

$$V = -\frac{GMm}{r}$$

Conservation of energy:

$$E = T + V = \frac{1}{2}mv^2 - \frac{GMm}{r} = const$$

At point A the mass starts the movement:

$$E_A = V_A = -\frac{GMm}{r_A}$$

$$\frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{r_A}$$

$$\Rightarrow v = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_A}\right)}$$

It is desired to find the course that minimizes the time function:

$$\min_{\theta} t = \int_{l} \frac{dl}{v} = \int_{r=r_A}^{r_B} \frac{\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}}{\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_A}\right)}} dr = \frac{1}{\sqrt{2GM}} \int_{r=r_A}^{r_B} \sqrt{\frac{1 + r^2\theta'(r)^2}{\frac{1}{r} - \frac{1}{r_A}}} dr$$

$$= \frac{1}{\sqrt{2GM}} \int_{r=r_A}^{r_B} \sqrt{\frac{(1 + r^2\theta'(r)^2)rr_A}{r_A - r}} dr$$

$$F(r, \theta, \theta'(r)) = \sqrt{\frac{(1 + r^2\theta'(r)^2)rr_A}{r_A - r}}$$

$$\frac{\partial F}{\partial \theta} - \frac{d}{dr}\frac{\partial F}{\partial \theta'} = 0$$

$$\frac{\partial F}{\partial \theta} = 0$$

$$\frac{\partial F}{\partial \theta'} = r^2\sqrt{\frac{rr_A}{r_A - r}}\frac{\theta'(r)}{\sqrt{1 + r^2\theta'(r)^2}}$$

$$\frac{\partial F}{\partial \theta} - \frac{d}{dr}\frac{\partial F}{\partial \theta'} = -\frac{d}{dr}\frac{\partial F}{\partial \theta'} = -\frac{d}{dr}\left(r^2\sqrt{\frac{rr_A}{r_A - r}}\frac{\theta'(r)}{\sqrt{1 + r^2\theta'(r)^2}}\right) = 0$$

$$\frac{\partial F}{\partial \theta'} = r^2\sqrt{\frac{rr_A}{r_A - r}}\frac{\theta'(r)}{\sqrt{1 + r^2\theta'(r)^2}} = const$$

Squaring both sides:

$$r^4\frac{rr_A}{r_A - r}\frac{\theta'(r)^2}{1 + r^2\theta'(r)^2} = c = const$$

$$\frac{r^5 r_A}{r_A - r}\theta'(r)^2 = c + cr^2\theta'(r)^2$$

$$\theta'(r)^2\left(\frac{r^5 r_A}{r_A - r} - cr^2\right) = c$$

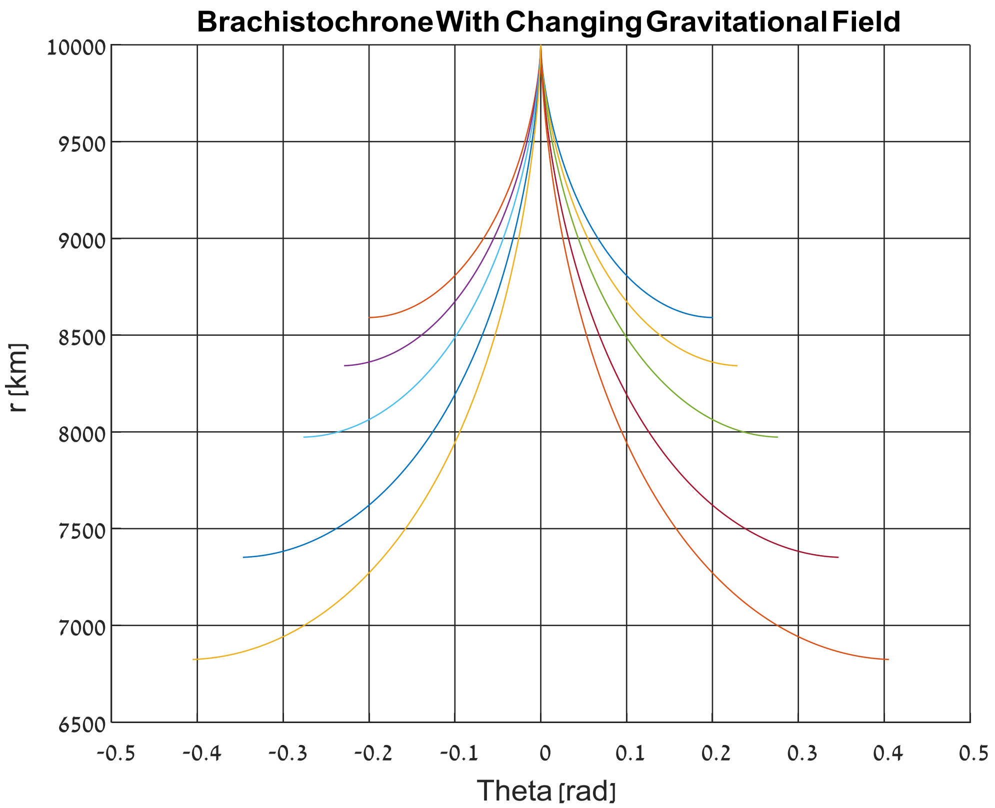
$$\theta'(r)^2 = \frac{c}{\frac{r^5 r_A}{r_A - r} - cr^2} = \frac{c(r_A - r)}{r^5 r_A - cr^2(r_A - r)}$$

$$\theta(r) = \pm \int_{r=r_A}^{r_B} \sqrt{\frac{c(r_A - r)}{r^5 r_A - cr^2(r_A - r)}}\,dr$$

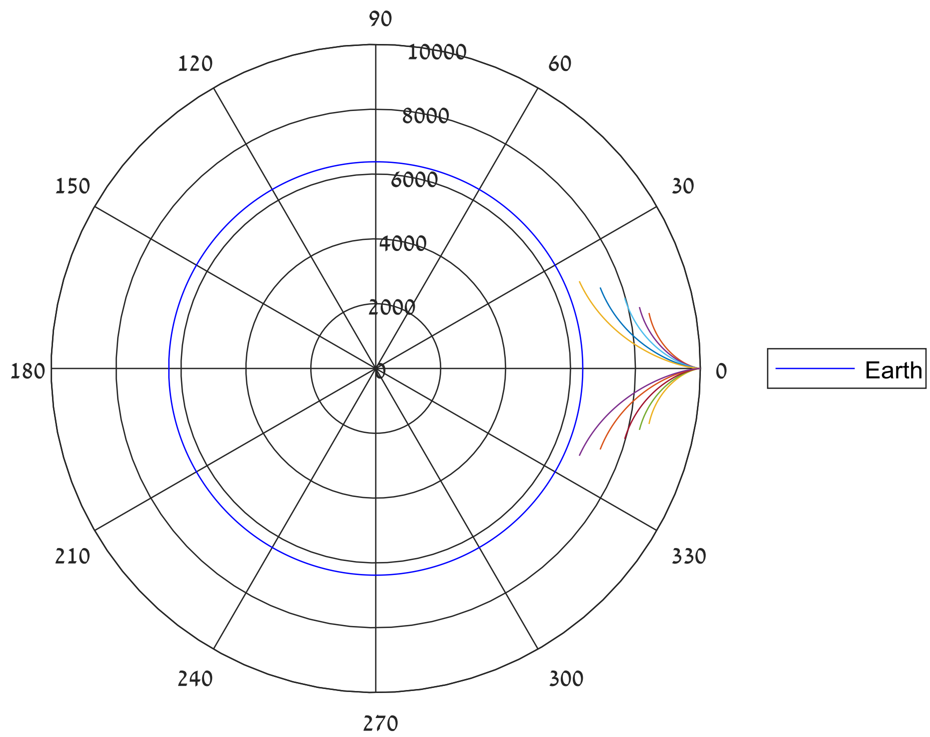
The initial and terminal conditions:

$$\theta(r_A) = \theta_A, \qquad \theta(r_B) = \theta_B$$

This is solved numerically:



Brachistochrone With Changing Gravitational Field



Bernoulli's Method:

The path of shortest time must satisfy Bouguer's law:

$$\frac{r\sin\phi}{v(r)} = const$$

ϕ is the angle between the tangent to the surface and the radius vector.

It was seen from energy conservation that v satisfies:

$$v = \sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_A}\right)}$$

Therefore:

$$\frac{r\sin\phi}{\sqrt{2GM\left(\frac{1}{r}-\frac{1}{r_A}\right)}} = const$$

$$\frac{r\sin\phi}{\sqrt{\frac{1}{r}-\frac{1}{r_A}}} = const$$

$$\frac{r\sin\phi}{\sqrt{\frac{r_A-r}{rr_A}}}=const$$

$$\sqrt{\frac{r^3r_A}{r_A-r}}\sin\phi=const$$

In order to find the path it is required to identify the relation between ϕ, and θ.

$$\sin\phi=\frac{rd\theta}{dl}$$

$$\sqrt{\frac{r^3r_A}{r_A-r}}\frac{rd\theta}{dl}=const=\sqrt{c}$$

Squaring both sides:

$$\frac{r^3r_A}{r_A-r}r^2d\theta^2=cdl^2=c[dr^2+r^2d\theta^2]$$

$$r^2d\theta^2\left(\frac{r^3r_A}{r_A-r}-c\right)=cdr^2$$

$$d\theta^2\left(\frac{r^5r_A}{r_A-r}-cr^2\right)=cdr^2$$

$$d\theta^2\left(\frac{r^5r_A-cr^2(r_A-r)}{r_A-r}\right)=cdr^2$$

$$\frac{d\theta^2}{dr^2}=\theta^{'}(r)^2=\frac{c(r_A-r)}{r^5r_A-cr^2(r_A-r)}$$

$$\theta(r)=\pm\int_{r=r_A}^{r_B}\sqrt{\frac{c(r_A-r)}{r^5r_A-cr^2(r_A-r)}}dr$$

The initial and terminal conditions:

$$\theta(r_A)=\theta_A,\qquad\theta(r_B)=\theta_B$$

The same expression has been obtained from both methods.

Solving the brachistichrone problem inside the earth

Assume a spherical earth with an internal gravitational field:

$$\underline{g} = -\frac{GM}{R^3} r\underline{\hat{r}}$$

The gravitational potential is derived to be:

$$V(r) = \frac{GMm}{2R}\left(\frac{r^2}{R^2} - 3\right)$$

The kinetic energy of a point mass during its course:

$$T = \frac{1}{2}mv^2$$

Conservation of energy:

$$E = T + V = \frac{1}{2}mv^2 + \frac{GMm}{2R}\left(\frac{r^2}{R^2} - 3\right) = const$$

At point A the mass starts the movement:

$$E_A = V_A = \frac{GMm}{2R}\left(\frac{{r_A}^2}{R^2} - 3\right)$$

$$\frac{1}{2}mv^2 + \frac{GMm}{2R}\left(\frac{r^2}{R^2} - 3\right) = \frac{GMm}{2R}\left(\frac{{r_A}^2}{R^2} - 3\right)$$

$$\frac{1}{2}mv^2 = \frac{GMm}{2R^3}({r_A}^2 - r^2)$$

$$\Rightarrow v = \sqrt{\frac{GM}{R^3}({r_A}^2 - r^2)}$$

$$dl = \sqrt{dr^2 + r^2 d\theta^2} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 d\theta^2 + r^2 d\theta^2} = \sqrt{r'(\theta)^2 + r^2}\, d\theta$$

It is desired to find the course that minimizes the time function:

$$\min_{\theta} t = \int_l \frac{dl}{v} = \int_{\theta=\theta_A}^{\theta_B} \frac{\sqrt{r'(\theta)^2 + r^2}}{\sqrt{\frac{GM}{R^3}({r_A}^2 - r^2)}} d\theta = \sqrt{\frac{R^3}{GM}} \int_{\theta=\theta_A}^{\theta_B} \frac{\sqrt{r'(\theta)^2 + r^2}}{\sqrt{{r_A}^2 - r^2}} d\theta$$

$$F\big(\theta, r, r'(\theta)\big) = \sqrt{\frac{r'(\theta)^2 + r^2}{{r_A}^2 - r^2}}$$

The Euler-Lagrange equations:

$$\frac{\partial F}{\partial r} - \frac{d}{d\theta}\frac{\partial F}{\partial r'} = 0$$

Since $\frac{\partial F}{\partial \theta} = 0$, Beltrami's identity is applied:

$$F - r'(\theta)\frac{\partial F}{\partial r'} = const$$

$$\frac{\partial F}{\partial r'} = \frac{r'(\theta)}{\sqrt{({r_A}^2 - r^2)(r'(\theta)^2 + r^2)}}$$

$$F - r'(\theta)\frac{\partial F}{\partial r'} = \sqrt{\frac{r'(\theta)^2 + r^2}{{r_A}^2 - r^2}} - \frac{r'(\theta)^2}{\sqrt{({r_A}^2 - r^2)(r'(\theta)^2 + r^2)}} = const$$

$$\frac{r'(\theta)^2 + r^2 - r'(\theta)^2}{\sqrt{({r_A}^2 - r^2)(r'(\theta)^2 + r^2)}} = \frac{r^2}{\sqrt{({r_A}^2 - r^2)(r'(\theta)^2 + r^2)}} = const$$

Squaring both sides:

$$\frac{r^4}{({r_A}^2 - r^2)(r'(\theta)^2 + r^2)} = c = const$$

$$\frac{r^4}{{r_A}^2 - r^2} = c(r'(\theta)^2 + r^2)$$

$$r'(\theta)^2 = \frac{r^4}{c({r_A}^2 - r^2)} - r^2 = \frac{r^4 - cr^2({r_A}^2 - r^2)}{c({r_A}^2 - r^2)}$$

A non linear differential equation has been received. A simple private solution may be obtained by assuming both points A, and B are of radius r_A. From the symmetry of the problem the path would have its minimum radius at the middle of the course:

$$r'(\theta_m) = 0, \qquad \theta_m = \frac{\theta_A + \theta_B}{2}$$

$$\Rightarrow c = \frac{{r_m}^2}{({r_A}^2 - {r_m}^2)}, \qquad r_m = r(\theta_m)$$

$$r'(\theta)^2 = \frac{r^4}{\frac{{r_m}^2}{({r_A}^2 - {r_m}^2)}({r_A}^2 - r^2)} - r^2 = \frac{r^4({r_A}^2 - {r_m}^2)}{{r_m}^2({r_A}^2 - r^2)} - r^2$$

$$= \frac{r^4({r_A}^2 - {r_m}^2) - r^2{r_m}^2({r_A}^2 - r^2)}{{r_m}^2({r_A}^2 - r^2)} = \frac{r^4{r_A}^2 - r^4{r_m}^2 - r^2{r_m}^2{r_A}^2 + r^4{r_m}^2}{{r_m}^2({r_A}^2 - r^2)}$$

$$= \frac{r^4{r_A}^2 - r^2{r_m}^2{r_A}^2}{{r_m}^2({r_A}^2 - r^2)} = \left(\frac{r_A r}{r_m}\right)^2 \frac{r^2 - {r_m}^2}{{r_A}^2 - r^2}$$

$$\frac{dr}{d\theta} = \pm \frac{r_A r}{r_m} \sqrt{\frac{r^2 - r_m{}^2}{r_A{}^2 - r^2}}$$

Rewriting in the form of $\theta(r)$ obtains an integral equation. Since the curve is symmetric θ can be integrated from the minimum point r_m to r

$$\frac{d\theta}{dr} = \pm \frac{r_m}{r_A r} \sqrt{\frac{r_A{}^2 - r^2}{r^2 - r_m{}^2}}$$

$$\theta(r) = \pm \frac{r_m}{r_A} \int_{r_m}^{r} \frac{1}{r} \sqrt{\frac{r_A{}^2 - r^2}{r^2 - r_m{}^2}} dr$$

The solution:

$$\theta(r) = \arctan\left(\frac{r_A}{r_m} \sqrt{\frac{r^2 - r_m{}^2}{r_A{}^2 - r^2}}\right) - \frac{r_m}{r_A} \arctan\left(\sqrt{\frac{r^2 - r_m{}^2}{r_A{}^2 - r^2}}\right) + c$$

$$\theta(r_m) = \arctan 0 - \frac{r_m}{r_A} \arctan 0 + c = c = \theta_m = \frac{\theta_A + \theta_B}{2}$$

$$\theta(r_A) = \arctan \infty - \frac{r_m}{r_A} \arctan \infty + c = \frac{\pi}{2} - \frac{\pi}{2} \frac{r_m}{r_A} + c = \theta_A$$

$$\frac{\pi}{2} - \frac{\pi}{2} \frac{r_m}{r_A} + \frac{\theta_A + \theta_B}{2} = \theta_A$$

$$\pi - \pi \frac{r_m}{r_A} + \theta_A + \theta_B = 2\theta_A$$

$$\theta_A - \theta_B = \pi - \pi \frac{r_m}{r_A}$$

This is the relation for the angular distance between points A, B, and the ratio between the initial radius and the minimum radius of the path.

Overall, the equation:

$$\theta(r) = \arctan\left(\frac{r_A}{r_m} \sqrt{\frac{r^2 - r_m{}^2}{r_A{}^2 - r^2}}\right) - \frac{r_m}{r_A} \arctan\left(\sqrt{\frac{r^2 - r_m{}^2}{r_A{}^2 - r^2}}\right) + \theta_m$$

While the end points:

$$\theta(r_A) = \theta_A, \qquad \theta(r_B = r_A) = \theta_B = \theta_A - \pi + \pi \frac{r_m}{r_A}$$

And the midpoint is:

$$\theta(r_m) = \theta_m = \frac{\theta_A + \theta_B}{2}$$

Bernoulli's Method:

The path of shortest time must satisfy Bouguer's law:

$$\frac{r \sin\phi}{v(r)} = const$$

ϕ is the angle between the tangent to the surface and the radius vector.

It was seen from energy conservation that v satisfies:

$$v = \sqrt{\frac{GM}{R^3}({r_A}^2 - r^2)}$$

Therefore:

$$\frac{r \sin\phi}{\sqrt{\frac{GM}{R^3}({r_A}^2 - r^2)}} = const$$

$$\frac{r \sin\phi}{\sqrt{{r_A}^2 - r^2}} = const$$

The relation between ϕ and θ:

$$\sin\phi = \frac{rd\theta}{dl}$$

$$\frac{r\frac{rd\theta}{dl}}{\sqrt{{r_A}^2 - r^2}} = const$$

Squaring both sides:

$$\frac{r^4}{{r_A}^2 - r^2}\frac{d\theta^2}{dl^2} = c = const$$

$$\frac{r^4}{{r_A}^2 - r^2}d\theta^2 = cdl^2 = c(dr^2 + r^2 d\theta^2)$$

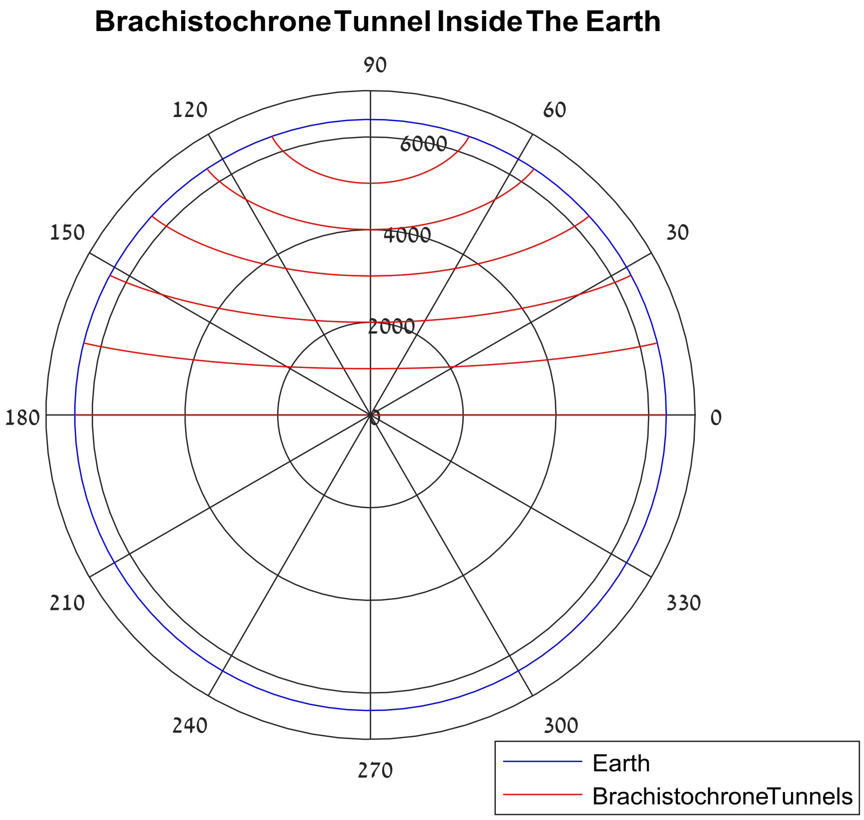
$$d\theta^2\left(\frac{r^4}{{r_A}^2 - r^2} - cr^2\right) = cdr^2$$

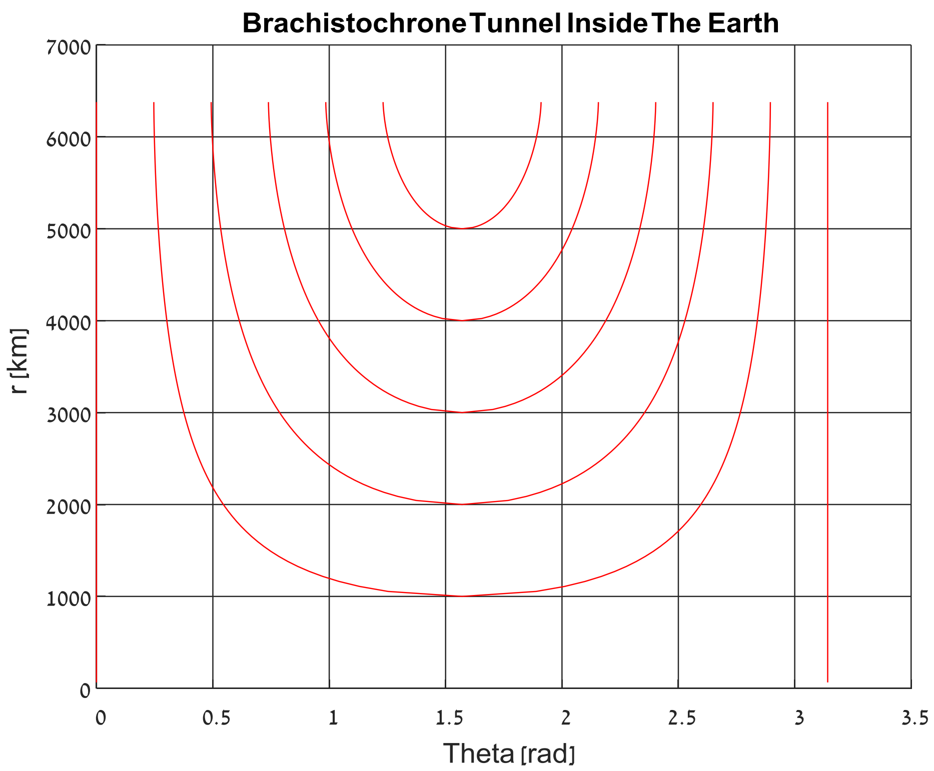
$$d\theta^2\left(r^4 - cr^2({r_A}^2 - r^2)\right) = c({r_A}^2 - r^2)dr^2$$

$$r'(\theta)^2 = \frac{dr^2}{d\theta^2} = \frac{r^4 - cr^2({r_A}^2 - r^2)}{c({r_A}^2 - r^2)}$$

$$r'(\theta)^2 = \frac{r^4}{c({r_A}^2 - r^2)} - r^2$$

The same expression has been obtained from both calculus of variations, and from Bernoulli's method.





Solving the brachistochrone problem with drag

Assume the constant gravity problem with a rigid body sliding on a surface with air resistance. the drag force applied on the mass:

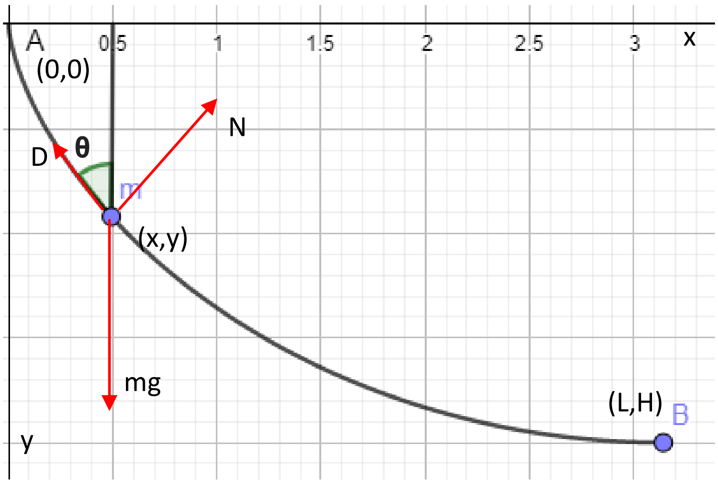
$$\underline{D} = -\frac{1}{2}\rho v^2 S C_D \hat{\underline{v}}$$

While ρ is the density of the air which is assumed constant for very low heights, S is the drag induced cross sectional area of the body which interacts with the air, C_D is the coefficient of drag of the body, assumed constant for low velocities.

Define:

$$k = \frac{\rho S C_D}{2m}$$

$$\underline{D} = -mkv^2\hat{\underline{v}}$$



It is assumed that the velocity vector coincides with the direction of motion on the surface path:

$$\underline{D} = -mkv^2 d\hat{\underline{l}}$$

The work of the drag force:

$$W_D = \int \underline{D}\cdot d\underline{l} = -\int mkv^2\, d\hat{\underline{l}}\cdot d\underline{l} = -\int mkv^2\, dl$$

There is no energy conservation, the energy loss is equal to the work done by the drag:

$$E = \frac{mv^2}{2} - mgy$$

$$dE = mvdv - mgdy = dW_D$$

$$mvdv - mgdy = -mkv^2 dl$$

$$v\frac{dv}{dx}dx - g\frac{dy}{dx}dx = -kv^2\sqrt{1+y'(x)^2}dx$$

$$v(x)v'(x) - gy'(x) + kv^2(x)\sqrt{1+y'(x)^2} = 0$$

A constraint on the path course has been received:

$$\Psi(x, y(x), y'(x), v(x), v'(x)) = v(x)v'(x) - gy'(x) + kv^2(x)\sqrt{1+y'(x)^2} = 0$$

$$v'(x) = \frac{g}{v(x)}y'(x) - kv(x)\sqrt{1+y'(x)^2}$$

While:

$$v(x_A = 0) = 0$$

A differential equation for v has been received.

It is required to find the path of minimal time, so the time of descent is our cost function:

$$\min_x t = \int_l \frac{dl}{v} = \int_{x=0}^{x=L} \frac{\sqrt{1+y'(x)^2}}{v(x)}dx$$

Adding the constraint to the cost function:

$$\min_x t = \int_{x=0}^{x=L} \left(\frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + kv^2(x)\sqrt{1+y'(x)^2}\right)\right)dx$$

Therefore:

$$H = \frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + kv^2(x)\sqrt{1+y'(x)^2}\right)$$

The Euler-Lagrange equations:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\frac{\partial H}{\partial y'} = 0, \qquad \frac{\partial H}{\partial v} - \frac{d}{dx}\frac{\partial H}{\partial v'} = 0$$

Since H is not explicitly dependant on x the Beltrami identity may be used:

$$H - y'(x)\frac{\partial H}{\partial y'} = c_1 = const, \qquad H - v'(x)\frac{\partial H}{\partial v'} = c_2 = const$$

$$\frac{\partial H}{\partial y'} = \frac{y'(x)}{v(x)\sqrt{y'(x)^2+1}} + \lambda(x)\left(-g + \frac{kv^2(x)y'(x)}{\sqrt{1+y'(x)^2}}\right)$$

$$= \frac{y'(x)}{v(x)\sqrt{y'(x)^2+1}} - g\lambda(x) + \frac{kv^2(x)y'(x)}{\sqrt{1+y'(x)^2}}\lambda(x)$$

$$H - y'(x)\frac{\partial H}{\partial y'} = \frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + kv^2(x)\sqrt{1+y'(x)^2}\right)$$
$$- \frac{y'(x)^2}{v(x)\sqrt{y'(x)^2+1}} + g\lambda(x)y'(x) - \frac{kv^2(x)y'(x)^2}{\sqrt{1+y'(x)^2}}\lambda(x) = c_1$$

$$\frac{1+y'(x)^2}{v(x)} + \lambda(x)\left(v(x)v'(x)\sqrt{1+y'(x)^2} + kv^2(x)(1+y'(x)^2)\right) - \frac{y'(x)^2}{v(x)}$$
$$- kv^2(x)y'(x)^2\lambda(x) = c_1\sqrt{1+y'(x)^2}$$

$$\frac{1}{v(x)} + \lambda(x)v(x)v'(x)\sqrt{1+y'(x)^2} + \lambda(x)kv^2(x) = c_1\sqrt{1+y'(x)^2}$$

$$\frac{1}{v(x)} + \lambda(x)kv^2(x) = \left(c_1 - \lambda(x)v(x)v'(x)\right)\sqrt{1+y'(x)^2}$$

$$1 + y'(x)^2 = \frac{\frac{1}{v(x)} + \lambda(x)kv^2(x)}{c_1 - \lambda(x)v(x)v'(x)}$$

$$y'(x) = \sqrt{\frac{1+\lambda(x)kv^3(x)}{c_1 v(x) - \lambda(x)v^2(x)v'(x)} - 1}$$

$$\frac{\partial H}{\partial v'} = \lambda(x)v(x)$$

$$H - v'(x)\frac{\partial H}{\partial v'} = \frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + kv^2(x)\sqrt{1+y'(x)^2}\right)$$
$$- \lambda(x)v(x)v'(x) = c_2$$

$$\frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(kv^2(x)\sqrt{1+y'(x)^2} - gy'(x)\right) = c_2$$

$$\lambda(x) = \frac{c_2 - \frac{\sqrt{1+y'(x)^2}}{v(x)}}{kv^2(x)\sqrt{1+y'(x)^2} - gy'(x)} = \frac{c_2 v(x) - \sqrt{1+y'(x)^2}}{kv^3(x)\sqrt{1+y'(x)^2} - gy'(x)v(x)}$$

Overall:

$$\begin{cases} y'(x) = \sqrt{\dfrac{1+\lambda(x)kv^3(x)}{c_1 v(x) - \lambda(x)v^2(x)v'(x)} - 1} \\ v'(x) = \dfrac{g}{v(x)}y'(x) - kv(x)\sqrt{1+y'(x)^2} \\ \lambda(x) = \dfrac{c_2 v(x) - \sqrt{1+y'(x)^2}}{kv^3(x)\sqrt{1+y'(x)^2} - gy'(x)v(x)} \end{cases}$$

There are two unknown constants and two integration constants, 4 constants overall.

Boundary conditions:

$$y(x_0 = 0) = 0, \qquad y(x_f = L) = H, \qquad v(x_0 = 0) = 0$$

Transversality condition, there is no constraint on $v(x_f)$:

$$\frac{\partial H}{\partial v'}(x_f = L) = 0$$

$$\lambda(x_f)v(x_f) = 0$$

Solving the Brachistochrone problem with varying density drag

Assume the constant gravity problem with a rigid body sliding on a surface with air resistance. The drag force applied on the mass:

$$\underline{D} = -\frac{1}{2}\rho v^2 S C_D \underline{\hat{v}} = -\frac{1}{2}\rho v^2 S C_D d\underline{\hat{l}}$$

The density is expressed by troposphere Pitot-statics model:

$$\rho = \rho_0 \left(1 - \frac{\beta_0 h}{T_0}\right)^{\frac{g}{R\beta_0}-1}$$

While the specific gas constant for air:

$$R = 287\left[\frac{J}{kg \cdot K}\right]$$

The temperature drop rate:

$$T(h) = T_0 - \beta_0 h, \qquad \beta_0 = 6.5 \cdot 10^{-3}\left[\frac{K}{m}\right]$$

Assuming STP conditions at sea level:

$$T_0 = 288[K], \qquad \rho_0 = 1.225\left[\frac{kg}{m^3}\right], \qquad P_0 = 101325[Pa]$$

Approximating as an exponential model for $\frac{\beta_0 h}{T_0} << 1$

$$\ln(1-x) \approx -x, \qquad x << 1$$

$$\rho = \rho_0 \left(1 - \frac{\beta_0 h}{T_0}\right)^{\frac{g}{R\beta_0}-1} = \rho_0 e^{\ln\left(\left(1-\frac{\beta_0 h}{T_0}\right)^{\frac{g}{R\beta_0}-1}\right)} = \rho_0 e^{\left(\frac{g}{R\beta_0}-1\right)\ln\left(1-\frac{\beta_0 h}{T_0}\right)} \approx \rho_0 e^{-\frac{\beta_0}{T_0}\left(\frac{g}{R\beta_0}-1\right)h}$$

$$= \rho_0 e^{-\frac{h}{H}}$$

While:

$$H = \frac{T_0}{\beta_0}\frac{1}{\frac{g}{R\beta_0}-1} = 10404[m]$$

Since the positive y direction is defined as negative altitude:

$$\rho = \rho_0 e^{\frac{y(x)}{H}}$$

Define:

$$k_0 = \frac{\rho_0 S C_D}{2m}$$

$$\underline{D} = -\frac{1}{2}\rho_0 e^{\frac{y(x)}{H}} v^2 S C_D d\hat{\underline{l}} = -m k_0 e^{\frac{y(x)}{H}} v^2 d\hat{\underline{l}}$$

There is no energy conservation, the energy loss is equal to the work done by the drag:

$$E = \frac{mv^2}{2} - mgy$$

$$dE = mvdv - mgdy = dW_D$$

$$mvdv - mgdy = -mk_0 e^{\frac{y(x)}{H}} v^2 dl$$

$$v\frac{dv}{dx}dx - g\frac{dy}{dx}dx = -k_0 e^{\frac{y(x)}{H}} v^2 \sqrt{1 + y'(x)^2}dx$$

$$\frac{dv}{dx} = \frac{g}{v}\frac{dy}{dx} - k_0 e^{\frac{y(x)}{H}} v\sqrt{1 + y'(x)^2}$$

While:

$$v(x_A = 0) = 0$$

A constraint on the path course has been received:

$$\Psi(x, y(x), y'(x), v(x), v'(x)) = v(x)v'(x) - gy'(x) + k_0 e^{\frac{y(x)}{H}} v^2(x)\sqrt{1 + y'(x)^2} = 0$$

It is required to find the path of minimal time, so the time of descent is our cost function:

$$\min_x t = \int_l \frac{dl}{v} = \int_{x=0}^{x=L} \frac{\sqrt{1 + y'(x)^2}}{v(x)} dx$$

Adding the constraint to the cost function:

$$\min_x t = \int_{x=0}^{x=L} \left(\frac{\sqrt{1 + y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + k_0 e^{\frac{y(x)}{H}} v^2(x)\sqrt{1 + y'(x)^2}\right) \right) dx$$

Therefore:

$$H = \frac{\sqrt{1 + y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + k_0 e^{\frac{y(x)}{H}} v^2(x)\sqrt{1 + y'(x)^2}\right)$$

The Euler-Lagrange equations:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\frac{\partial H}{\partial y'} = 0, \qquad \frac{\partial H}{\partial v} - \frac{d}{dx}\frac{\partial H}{\partial v'} = 0$$

Since H is not explicitly dependant on x the Beltrami identity may be used:

$$H - y'(x)\frac{\partial H}{\partial y'} = c_1 = const, \qquad H - v'(x)\frac{\partial H}{\partial v'} = c_2 = const$$

$$\frac{\partial H}{\partial y'} = \frac{y'(x)}{v(x)\sqrt{y'(x)^2+1}} + \lambda(x)\left(-g + \frac{k_0 e^{\frac{y(x)}{H}} v^2(x) y'(x)}{\sqrt{1+y'(x)^2}}\right)$$

$$= \frac{y'(x)}{v(x)\sqrt{y'(x)^2+1}} - g\lambda(x) + \frac{k_0 e^{\frac{y(x)}{H}} v^2(x) y'(x)}{\sqrt{1+y'(x)^2}}\lambda(x)$$

$$H - y'(x)\frac{\partial H}{\partial y'} = \frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + k_0 e^{\frac{y(x)}{H}} v^2(x)\sqrt{1+y'(x)^2}\right)$$
$$- \frac{y'(x)^2}{v(x)\sqrt{y'(x)^2+1}} + g\lambda(x)y'(x) - \frac{k_0 e^{\frac{y(x)}{H}} v^2(x) y'(x)^2}{\sqrt{1+y'(x)^2}}\lambda(x) = c_1$$

$$\frac{1+y'(x)^2}{v(x)} + \lambda(x)\left(v(x)v'(x)\sqrt{1+y'(x)^2} + k_0 e^{\frac{y(x)}{H}} v^2(x)(1+y'(x)^2)\right) - \frac{y'(x)^2}{v(x)}$$
$$- k_0 e^{\frac{y(x)}{H}} v^2(x) y'(x)^2 \lambda(x) = c_1\sqrt{1+y'(x)^2}$$

$$\frac{1}{v(x)} + \lambda(x)v(x)v'(x)\sqrt{1+y'(x)^2} + \lambda(x)k_0 e^{\frac{y(x)}{H}} v^2(x) = c_1\sqrt{1+y'(x)^2}$$

$$\frac{1}{v(x)} + \lambda(x)k_0 e^{\frac{y(x)}{H}} v^2(x) = \left(c_1 - \lambda(x)v(x)v'(x)\right)\sqrt{1+y'(x)^2}$$

$$1 + y'(x)^2 = \frac{\frac{1}{v(x)} + \lambda(x)k_0 e^{\frac{y(x)}{H}} v^2(x)}{c_1 - \lambda(x)v(x)v'(x)}$$

$$y'(x) = \sqrt{\frac{1 + \lambda(x)k_0 e^{\frac{y(x)}{H}} v^3(x)}{c_1 v(x) - \lambda(x)v^2(x)v'(x)} - 1}$$

$$\frac{\partial H}{\partial v'} = \lambda(x)v(x)$$

$$H - v'(x)\frac{\partial H}{\partial v'} = \frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(v(x)v'(x) - gy'(x) + k_0 e^{\frac{y(x)}{H}} v^2(x)\sqrt{1+y'(x)^2}\right)$$
$$- \lambda(x)v(x)v'(x) = c_2$$

$$\frac{\sqrt{1+y'(x)^2}}{v(x)} + \lambda(x)\left(k_0 e^{\frac{y(x)}{H}} v^2(x)\sqrt{1+y'(x)^2} - gy'(x)\right) = c_2$$

$$\lambda(x) = \frac{c_2 - \frac{\sqrt{1+y'(x)^2}}{v(x)}}{k_0 e^{\frac{y(x)}{H}} v^2(x)\sqrt{1+y'(x)^2} - gy'(x)} = \frac{c_2 v(x) - \sqrt{1+y'(x)^2}}{k_0 e^{\frac{y(x)}{H}} v^3(x)\sqrt{1+y'(x)^2} - gy'(x)v(x)}$$

Overall:

$$\begin{cases} y'(x) = \sqrt{\dfrac{1 + \lambda(x) k_0 e^{\frac{y(x)}{H}} v^3(x)}{c_1 v(x) - \lambda(x) v^2(x) v'(x)} - 1} \\ v'(x) = \dfrac{g}{v(x)} y'(x) - k_0 e^{\frac{y(x)}{H}} v(x) \sqrt{1 + y'(x)^2} \\ \lambda(x) = \dfrac{c_2 v(x) - \sqrt{1 + y'(x)^2}}{k_0 e^{\frac{y(x)}{H}} v^3(x) \sqrt{1 + y'(x)^2} - g y'(x) v(x)} \end{cases}$$

There are two unknown constants and two integration constants, 4 constants overall.

Boundary conditions:

$$y(x_0 = 0) = 0, \qquad y(x_f = L) = H, \qquad v(x_0 = 0) = 0$$

Transversality condition, there is no constraint on $v(x_f)$:

$$\frac{\partial H}{\partial v'}(x_f = L) = 0$$

$$\lambda(x_f) v(x_f) = 0$$

Solving the Brachistochrone problem for a spherical earth with drag

Assume a spherical earth with a gravitational field:

$$\underline{g} = -\frac{M_{\oplus}G}{r^2}\underline{\hat{r}}$$

A rigid body sliding on a surface with air resistance. The drag force applied on the mass:

$$\underline{D} = -\frac{1}{2}\rho v^2 S C_D \underline{\hat{v}} = -\frac{1}{2}\rho v^2 S C_D d\underline{\hat{l}}$$

The density model:

$$\rho = \rho_0 e^{-\frac{h}{H}}$$

At polar coordinates the altitude is:

$$h = r - R_{\oplus}$$

So, the density model:

$$\rho = \rho_0 e^{\frac{R_{\oplus}-r}{H}}$$

Define:

$$k_0 = \frac{\rho_0 S C_D}{2m}$$

$$\underline{D} = -\frac{1}{2}\rho_0 e^{\frac{R_{\oplus}-r}{H}} v^2 S C_D d\underline{\hat{l}} = -m k_0 e^{\frac{R_{\oplus}-r}{H}} v^2 d\underline{\hat{l}}$$

There is no energy conservation, the energy loss is equal to the work done by the drag:

$$E = \frac{mv^2}{2} - \frac{GMm}{r}$$

$$dE = mvdv + \frac{GMm}{r^2}dr = dW_D$$

$$mvdv + \frac{GMm}{r^2}dr = -mk_0 e^{\frac{R_{\oplus}-r}{H}} v^2 dl$$

$$v\frac{dv}{d\theta}d\theta + \frac{GM}{r^2}dr = -k_0 e^{\frac{R_{\oplus}-r}{H}} v^2 \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}d\theta$$

$$\frac{dv}{d\theta} = -\frac{GM}{vr^2}\frac{dr}{d\theta} - k_0 e^{\frac{R_{\oplus}-r}{H}} v\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

While:

$$v(\theta_A = 0) = 0$$

A constraint on the path course has been received:

$$\Psi(\theta, r(\theta), r'(\theta), v(\theta), v'(\theta)) = vv'(\theta) + \frac{GM}{r^2} r'(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^2 \sqrt{r'(\theta)^2 + r^2} = 0$$

It is desired to find the course that minimizes the time function:

$$\min_\theta t = \int_l \frac{dl}{v} = \int_{\theta=\theta_A}^{\theta_B} \frac{\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}}{v(\theta)} d\theta$$

Adding the constraint to the cost function:

$$\min_\theta t = \int_{\theta=\theta_A}^{\theta_B} \left(\frac{\sqrt{r'(\theta)^2 + r^2}}{v(\theta)} + \lambda(\theta) \left(vv'(\theta) + \frac{GM}{r^2} r'(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^2 \sqrt{r'(\theta)^2 + r^2} \right) \right) dx$$

Therefore:

$$H = \frac{\sqrt{r'(\theta)^2 + r^2}}{v(\theta)} + \lambda(\theta) \left(vv'(\theta) + \frac{GM}{r^2} r'(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^2 \sqrt{r'(\theta)^2 + r^2} \right)$$

The Euler-Lagrange equations:

$$\frac{\partial H}{\partial r} - \frac{d}{d\theta} \frac{\partial H}{\partial r'} = 0, \qquad \frac{\partial H}{\partial v} - \frac{d}{d\theta} \frac{\partial H}{\partial v'} = 0$$

Since H is not explicitly dependant on θ the Beltrami identity may be used:

$$H - r'(\theta) \frac{\partial H}{\partial r'} = c_1 = const, \qquad H - v'(\theta) \frac{\partial H}{\partial v'} = c_2 = const$$

$$\frac{\partial H}{\partial r'} = \frac{r'(\theta)}{v(\theta)\sqrt{r'(\theta)^2 + r^2}} + \lambda(\theta) \frac{GM}{r^2} + \lambda(\theta) \frac{k_0 e^{\frac{R_\oplus - r}{H}} v^2 r'(\theta)}{\sqrt{r'(\theta)^2 + r^2}}$$

$$H - r'(\theta) \frac{\partial H}{\partial r'} = \frac{\sqrt{r'(\theta)^2 + r^2}}{v(\theta)} + \lambda(\theta) \left(vv'(\theta) + \frac{GM}{r^2} r'(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^2 \sqrt{r'(\theta)^2 + r^2} \right)$$
$$- \frac{r'(\theta)^2}{v(\theta)\sqrt{r'(\theta)^2 + r^2}} - \lambda(\theta) \frac{GM}{r^2} r'(\theta) - \lambda(\theta) \frac{k_0 e^{\frac{R_\oplus - r}{H}} v^2 r'(\theta)^2}{\sqrt{r'(\theta)^2 + r^2}} = c_1$$

$$\frac{r'(\theta)^2 + r^2}{v(\theta)} + \lambda(\theta) vv'(\theta) \sqrt{r'(\theta)^2 + r^2} + \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^2 (r'(\theta)^2 + r^2) - \frac{r'(\theta)^2}{v(\theta)}$$
$$- \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^2 r'(\theta)^2 = c_1 \sqrt{r'(\theta)^2 + r^2}$$

$$\frac{r^2}{v(\theta)} + \lambda(\theta) vv'(\theta) \sqrt{r'(\theta)^2 + r^2} + \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^2 r^2 = c_1 \sqrt{r'(\theta)^2 + r^2}$$

$$\frac{r^2}{v(\theta)} + \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^2 r^2 = \left(c_1 - \lambda(\theta) vv'(\theta)\right) \sqrt{r'(\theta)^2 + r^2}$$

$$\sqrt{r'(\theta)^2 + r^2} = \frac{\frac{r^2}{v(\theta)} + \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^2 r^2}{c_1 - \lambda(\theta) v v'(\theta)}$$

$$r'(\theta)^2 + r^2 = \left(\frac{r^2 + \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^3(\theta) r^2}{c_1 v(\theta) - \lambda(\theta) v^2(\theta) v'(\theta)} \right)^2$$

$$r'(\theta) = \sqrt{\left(\frac{r^2 + \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^3(\theta) r^2}{c_1 v(\theta) - \lambda(\theta) v^2(\theta) v'(\theta)} \right)^2 - r^2}$$

$$\frac{\partial H}{\partial v'} = \lambda(\theta) v(\theta)$$

$$H - v'(\theta) \frac{\partial H}{\partial v'} = \frac{\sqrt{r'(\theta)^2 + r^2}}{v(\theta)} + \lambda(\theta) \left(v v'(\theta) + \frac{GM}{r^2} r'(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^2 \sqrt{r'(\theta)^2 + r^2} \right) - \lambda(\theta) v(\theta) v'(\theta) = c_2$$

$$\frac{\sqrt{r'(\theta)^2 + r^2}}{v(\theta)} + \lambda(\theta) \left(\frac{GM}{r^2} r'(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^2 \sqrt{r'(\theta)^2 + r^2} \right) = c_2$$

$$\lambda(\theta) = \frac{c_2 - \frac{\sqrt{r'(\theta)^2 + r^2}}{v(\theta)}}{\frac{GM}{r^2} r'(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^2 \sqrt{r'(\theta)^2 + r^2}} = \frac{c_2 v(\theta) - \sqrt{r'(\theta)^2 + r^2}}{\frac{GM}{r^2} r'(\theta) v(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^3(\theta) \sqrt{r'(\theta)^2 + r^2}}$$

Overall:

$$\begin{cases} r'(\theta) = \sqrt{\left(\frac{r^2 + \lambda(\theta) k_0 e^{\frac{R_\oplus - r}{H}} v^3(\theta) r^2}{c_1 v(\theta) - \lambda(\theta) v^2(\theta) v'(\theta)} \right)^2 - r^2} \\ v'(\theta) = -\frac{GM}{v(\theta) r^2(\theta)} r'(\theta) - k_0 e^{\frac{R_\oplus - r(\theta)}{H}} v(\theta) \sqrt{r'(\theta)^2 + r^2(\theta)} \\ \lambda(\theta) = \frac{c_2 v(\theta) - \sqrt{r'(\theta)^2 + r^2}}{\frac{GM}{r^2} r'(\theta) v(\theta) + k_0 e^{\frac{R_\oplus - r}{H}} v^3(\theta) \sqrt{r'(\theta)^2 + r^2}} \end{cases}$$

There are two unknown constants and two integration constants, 4 constants overall.

Boundary conditions:

$$r(\theta_A) = r_A, \qquad r(\theta_B) = r_B, \qquad v(\theta_A) = 0$$

Transversality condition, there is no constraint on $v(\theta_B)$:

$$\frac{\partial H}{\partial v'}(\theta_B) = 0$$

$$\lambda(\theta_B) v(\theta_B) = 0$$

Conclusion

This paper presented and discussed the Brachistochrone problem, defined the statement of the problem by Johan Bernoulli in 1696. In order to solve the problem and to find the shortest path the parametrisation equations of the cycloid curve were computed from a rolling circle on a straight line, Snell's law was derived using Fermat's principle of minimal time. Also, there was a brief introduction to calculus of variations and the tools which were used in the proof were presented. Afterwards, the Brachistochrone problem was solved using Bernoulli's method of analogy to light, and by variational calculus method. The time of descent was computed for the cycloid curve and it was received that the time to reach the bottom of the cycloid is the same with no regard where the point mass is placed. Lastly, the time of descent of a straight line was computed and compared with cycloid to confirm that the time of descent of the cycloid is indeed the minimal possible time. In the second part of the report the problem was generalized using several realistic influences and their effect on the Brachistochrone curve was derived and analyzed. The gravitational potential of a perfect sphere was derived both inside, and outside the sphere, and using it the Brachistochrone problem was solved for round earth solutions via Bernoulli's method and with calculus of variations. It was seen that both methods provided the same exact solution so they verified each over. Afterwards the problem was solved including quadratic drag forces, for low heights with constant drag, for varying density and lastly for a round earth. The solutions for the drag problems were received as a set of algebraic differential equations, and solving it requires advanced numerical integration methods.

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